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Unsteady fully-developed flow in a curved pipe

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Abstract. It is shown that the boundary layer which develops from rest in a loosely coiled pipe of circular cross-section, following the imposition of a constant pressure gradient, terminates in singular behaviour at the inside bend after a finite time. This singularity of the boundary-layer equations is interpreted as an eruption of boundary-layer fluid into the interior or core flow. This result complements earlier work by Stewartson *et al.* [1] who consider the steady inlet flow to a curved pipe at high Dean number. In that case a singularity also develops, now at a finite distance from the entrance at the inside bend, which is again interpreted in terms of a boundary-layer collision or eruption.

Keywords: curved-pipe flow, high Dean number, unsteady boundary layer, boundary-layer eruption, singularity.

1. Introduction

The flow in a curved pipe, whether laminar or turbulent, is of importance, for example, in problems as diverse as heat exchangers and the aorta. Fully-developed steady flow in terms of small values of a parameter that characterises the flow, and which bears his name, was first studied by Dean [2, 3]. It is the flow at high Dean number that has attracted most recent attention however, and which is of concern to us here. Accurate and extensive calculations of the steady, laminar, fully-developed flow in a loosely coiled pipe of circular cross-section, using the full equations, have been carried out by Collins and Dennis [4], and Dennis [5]. From these, as the Dean number increases, an asymptotic structure emerges in which an inviscid interior, or core, flow is flanked by a boundary layer at the pipe wall. Approximate theories, at high Dean number, based on this idea have been presented by Barua [6] and Ito [7]. However such asymptotic theories have never been made precise and fully consistent. The relationship between asymptotic theory and the full results has been surveyed by Dennis and Riley [8] and, in particular, the technical difficulties associated with the former have been outlined.

In this paper we consider an unsteady problem associated with the configuration under consideration, namely the fully-developed laminar flow when a constant pressure gradient along the pipe is suddenly imposed. By fully developed we imply that flow conditions are independent of distance measured axially along the pipe. This problem has previously been considered by Lam [9], the major difference is that Lam works throughout with a *Lagrangian* flow description, see Van Dommelen and Cowley [10] for details of this approach, whilst we adopt a more conventional Eulerian formulation. Attention should also be drawn to earlier work by Farthing [11] who considers the initial development in time by a series approach (see also Pedley [12]).

We identify time scales on which the core and boundary-layer flows develop. The latter is the shorter, and is the time scale adopted. The investigation of an unsteady flow complements

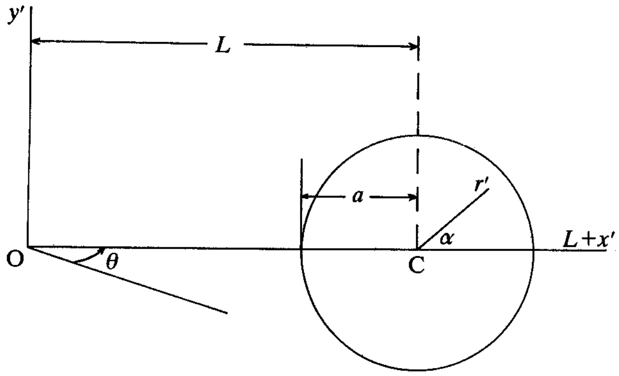

Figure 1. Definition sketch.

that of Stewartson *et al.* [1] who consider the steady entry flow to a curved pipe. In that case there are two axial length scales involved, and the shorter is the scale on which the boundary layer develops. In both [1] and here the axial core flow is taken to be uniform. Our study focuses on the flow at the inside bend of the pipe; by contrast Lam [9] traces the boundary layer from its origins at the outside bend up to the inside bend. The main feature of the flow is that after a finite time, the boundary-layer solution develops a singularity which, in the context of the flow overall, may be interpreted as the manifestation of an eruption of the boundary-layer fluid into the core. This in turn will lead to a modification of the core flow as it evolves towards a steady state.

Singular behaviour at the inside bend in the steadily developing flow [1] is also encountered, at a finite distance downstream from the inlet, with the same interpretation. However, the singularity structure differs in the two cases. For example, in the present case the axial shear stress remains finite, whilst in [1] it vanishes, suggesting some form of axial flow separation. The structure of the singularity, discussed in Section 4 below, is of the form introduced by Banks and Zaturska [13] in their study of the flow at the equator of a rotating sphere started from rest. Both the flows considered here, and in [13], are essentially two-dimensional in their terminal stages, which suggests that the singular behaviour uncovered in [13] has wide applicability. We conclude by discussing the implications of our present work, and that of [1], for the structure of the fully-developed steady flow at high Dean number.

2. Equations of motion

For the problem under consideration, namely the fully-developed unsteady flow in a loosely coiled pipe of circular cross-section ($a/L \ll 1$, see Figure 1), we may write the dimensionless equations for incompressible flow as, neglecting terms of relative order a/L,

$$\frac{1}{T}\frac{\partial w}{\partial t} + \frac{1}{r}\left(-\frac{\partial \phi}{\partial r}\frac{\partial w}{\partial \alpha} + \frac{\partial \phi}{\partial \alpha}\frac{\partial w}{\partial r}\right) = D + \nabla^2 w, \tag{2.1}$$

$$\nabla^4\phi + \frac{1}{r}\left(\frac{\partial \phi}{\partial r}\frac{\partial}{\partial \alpha} - \frac{\partial \phi}{\partial \alpha}\frac{\partial}{\partial r}\right)\nabla^2\phi + w\left(\sin\alpha\frac{\partial w}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{T}\frac{\partial}{\partial t}(\nabla^2\phi) = 0. \tag{2.2}$$

In these equations lengths are scaled with a, the axial velocity w has been scaled with ν $(L/2a^3)^{\frac{1}{2}}$, the stream function ϕ with ν and time t with a typical time t_0 to be chosen. The transverse and radial components of velocity in the cross-flow plane are given, respectively, by

$$u = -\frac{\partial \phi}{\partial r}, \qquad v = \frac{1}{r}\frac{\partial \phi}{\partial \alpha}. \tag{2.3}$$

The dimensionless parameters D, T in the above are defined as

$$D = Ga^3(2a/L)^{\frac{1}{2}}/\rho\nu^2, \qquad T = \nu t_0/a^2, \tag{2.4}$$

where $G = L^{-1}\partial p/\partial\theta$ is the constant axial pressure gradient. The parameter D is a form of the Dean number, in particular that adopted by Collins and Dennis [4] where its relationship to an alternative form used by some authors is discussed.

For fully-developed steady flow Dennis and Riley [8] following Ito [7] have argued that in the core, outside any boundary layers that form when $D \gg 1$, $\phi = O(D^{\frac{1}{3}})$, $w = O(D^{\frac{2}{3}})$; these scales are supported by the numerical solutions of the full Navier-Stokes equations presented by Collins and Dennis [4]. A time scale for the transverse motion in the core region is then a/V_t, where $V_t = O(\nu D^{\frac{1}{3}}/a)$, so that $t_0 = a^2 D^{-\frac{1}{3}}/\nu = t_c$ say. On this time scale, with $\phi = D^{\frac{1}{3}}\phi_c$, $w = D^{\frac{2}{3}} w_c$ and $D \gg 1$, Equations (2.1) and (2.2) become, at leading order,

$$\frac{\partial w_c}{\partial t} + \frac{1}{r}\left(-\frac{\partial \phi_c}{\partial r}\frac{\partial w_c}{\partial \alpha} + \frac{\partial \phi_c}{\partial \alpha}\frac{\partial w_c}{\partial r}\right) = 1, \tag{2.5}$$

$$w_c\left(\sin\alpha\frac{\partial w_c}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial w_c}{\partial \alpha}\right) = 0. \tag{2.6}$$

With $w_c \neq 0$, and introducing the Cartesian co-ordinates of Figure 1, Equations (2.5) and (2.6) become

$$\frac{\partial w_c}{\partial t} + \frac{\partial \phi_c}{\partial y}\frac{\partial w_c}{\partial x} = 1, \qquad \frac{\partial w_c}{\partial y} = 0, \tag{2.7, 2.8}$$

from which we deduce that

$$w_c = f(x,t), \qquad \phi_c = \frac{1-f_t}{f_x}y + g(x,t),$$

where f, g are arbitrary.

Consider next the boundary layer at $r = 1$ associated with this core flow. If we write $r = 1 - D^{-\frac{1}{3}}\zeta$, $w = D^{\frac{2}{3}}\bar{w}$, $\phi = D^{\frac{1}{3}}\bar{\phi}$ then, with

$$\overline{u} = \frac{\partial \overline{\phi}}{\partial \zeta}, \qquad \overline{v} = -\frac{\partial \overline{\phi}}{\partial \alpha}, \tag{2.9}$$

so that $\overline{v}$ is measured in the direction of ζ increasing, Equation (2.1) becomes, at leading order,

$$\frac{a^2}{\nu t_0}D^{-\frac{2}{3}}\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial \alpha} + \overline{v}\frac{\partial \overline{w}}{\partial \zeta} = \frac{\partial^2 \overline{w}}{\partial \zeta^2}. \tag{2.10}$$

Equation (2.10) indicates that an appropriate time scale for the developing boundary layer is $t_0 = a^2 D^{-\frac{2}{3}}/\nu = t_b$ say. We see then that $t_b = D^{-\frac{1}{3}} t_c$, which shows that changes to the core flow take place on a time scale which is much longer than the time scale for changes in the boundary layer. On the shorter time scale t_b, Equation (2.8) is unchanged to give $w_c = f(x, t)$, but (2.7) now shows that f is independent of t. With an impulsively applied pressure gradient at $t = 0$ we take, without loss of generality, $w_c \equiv 1$. It is of interest at this stage to make a comparison with the work of Stewartson, Cebeci and Chang [1]. They consider the steady developing flow in a loosely coiled pipe with a uniform axial flow on entry. Consistent with the above, it is found that changes within the boundary layer take place on an axial length scale much smaller than that on which changes in the core take place, which leads to $w_c \equiv 1$ for the purposes of the boundary-layer calculation.

On the time scale t_b we may write our boundary-layer equations, integrating the boundary-layer form of (2.2) once and introducing the velocity components (2.9), as

$$\frac{\partial \overline{u}}{\partial \alpha} + \frac{\partial \overline{v}}{\partial \zeta} = 0, \tag{2.11}$$

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial \alpha} + \overline{v}\frac{\partial \overline{u}}{\partial \zeta} + \tfrac{1}{2} \sin \alpha (\overline{w}^2 - 1) = \frac{\partial^2 \overline{u}}{\partial \zeta^2}, \tag{2.12}$$

$$\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial \alpha} + \overline{v}\frac{\partial \overline{w}}{\partial \zeta} = \frac{\partial^2 \overline{w}}{\partial \zeta^2}. \tag{2.13}$$

The boundary conditions that must be satisfied, for a flow started from rest, are as follows:

$$\left.\begin{array}{l} \overline{u} = \overline{v} = \overline{w} = 0, \quad \text{at} \quad \zeta = 0, \quad t \geqslant 0, \\ \overline{u} \to 0, \quad \overline{w} \to 1 \quad \text{as} \quad \zeta \to \infty, \quad t \geqslant 0, \\ \overline{u} = 0, \quad \overline{w} = 1 \quad \text{at} \quad t = 0, \quad \zeta > 0. \end{array}\right\} \tag{2.14}$$

3. Solution procedure

At the initial instant a vortex sheet is created at $\zeta = 0$. To accommodate this singular behaviour, and resolve the structure of the growing boundary layer for small t, it proves convenient to introduce new time and space co-ordinates as

$$\tau = \frac{t}{1+t}, \qquad \eta = \frac{\zeta}{\tau^{\frac{1}{2}}} = \zeta\frac{(1+t)^{\frac{1}{2}}}{t^{\frac{1}{2}}}, \tag{3.1}$$

so that Equations (2.11) to (2.13) become

$$\tau^{\frac{1}{2}}\frac{\partial \overline{u}}{\partial \alpha} + \frac{\partial \overline{v}}{\partial \eta} = 0, \tag{3.2}$$

$$(1-\tau)^2\left(\frac{\partial \overline{u}}{\partial \tau} - \frac{\eta}{2\tau}\frac{\partial \overline{u}}{\partial \eta}\right) + \overline{u}\frac{\partial \overline{u}}{\partial \alpha} + \tau^{-\frac{1}{2}}\overline{v}\frac{\partial \overline{u}}{\partial \eta} + \tfrac{1}{2} \sin \alpha (\overline{w}^2 - 1) = \tau^{-1}\frac{\partial^2 \overline{u}}{\partial \eta^2}, \tag{3.3}$$

$$(1-\tau)^2\left(\frac{\partial\overline{w}}{\partial\tau}-\frac{\eta}{2\tau}\frac{\partial\overline{w}}{\partial\eta}\right)+\overline{u}\frac{\partial\overline{w}}{\partial\alpha}+\tau^{-\frac{1}{2}}\overline{v}\frac{\partial\overline{w}}{\partial\eta}=\tau^{-1}\frac{\partial^2\overline{w}}{\partial\eta^2}, \tag{3.4}$$

together with

$$\left.\begin{array}{l}\overline{u}=\overline{v}=\overline{w}=0, \quad \text{at} \quad \eta=0, \quad t\geqslant 0,\\ \overline{u}\to 0, \quad \overline{w}\to 1 \quad \text{as} \quad \eta\to\infty, \quad \tau\geqslant 0,\\ \overline{u}=0, \quad \overline{w}=1 \quad \text{at} \quad \tau=0, \quad \eta>0.\end{array}\right\} \tag{3.5}$$

The solution at $\tau=0$ is simply $\overline{u}=\overline{v}=0$, $\overline{w}=\mathrm{erf}(\frac{1}{2}\eta)$, which provides the initial solution for a time-marching numerical solution of equations (3.2) to (3.5).

To carry out an integration, in time, of the above equations for all α and η is a formidable task. However, it is the solution at the inside bend, $\alpha=\pi$, that is of greatest interest to us. As we discuss in Section 5 below, it is this point that appears to cause problems in any steady solution of (2.11) to (2.14) and, furthermore, the work of Stewartson *et al.* [1] shows that an eruption of fluid takes place at $\alpha=\pi$ at a finite axial distance from the inlet in the steady developing flow. This eruption manifests itself as a singularity in the solution along $\alpha=\pi$.

To analyse the solution close to $\alpha=\pi$ we write

$$\begin{aligned}\overline{u} &= (\pi-\alpha)U(\eta,\tau)+O(\pi-\alpha)^3,\\ \overline{v} &= V(\eta,\tau)+O(\pi-\alpha)^2, \overline{w}=W(\eta,\tau)+O(\pi-\alpha)^2,\end{aligned} \tag{3.6}$$

where, from (3.2) to (3.5), U, V, W satisfy

$$-\tau^{\frac{1}{2}}U+\frac{\partial V}{\partial\eta}=0, \tag{3.7}$$

$$(1-\tau)^2\left(\tau\frac{\partial U}{\partial\tau}-\frac{\eta}{2}\frac{\partial U}{\partial\eta}\right)-\tau U^2+\tau^{\frac{1}{2}}V\frac{\partial U}{\partial\eta}+\frac{\tau}{2}(W^2-1)=\frac{\partial^2 U}{\partial\eta^2}, \tag{3.8}$$

$$(1-\tau)^2\left(\tau\frac{\partial W}{\partial\tau}-\frac{\eta}{2}\frac{\partial W}{\partial\eta}\right)+\tau^{\frac{1}{2}}V\frac{\partial W}{\partial\eta}=\frac{\partial^2 W}{\partial\eta^2}, \tag{3.9}$$

together with

$$\left.\begin{array}{l}U=V=W=0 \quad \text{at} \quad \eta=0, \quad \tau\geqslant 0\\ U\to 0, \quad W\to 1 \quad \text{as} \quad \eta\to\infty, \quad \tau\geqslant 0\end{array}\right\} \tag{3.10}$$

and the initial solution, at $\tau=0$,

$$U=V=0, \qquad W=\mathrm{erf}(\tfrac{1}{2}\eta). \tag{3.11}$$

Since we may expect $U\geqslant 0$ for $\alpha<\pi$, the solution close to $\alpha=\pi$ cannot be uninfluenced by conditions in $\alpha<\pi$. However, in this as in other similar problems, for example that considered by Stewartson *et al.* [1], as a singular behaviour of the solution is approached, which in the case under consideration we interpret as an eruption of fluid from the boundary layer into the core, the essential character of the solution will be uninfluenced by conditions in $\alpha < \pi$.

To solve Equations (3.7) to (3.9) subject to (3.10), with (3.11) as an initial solution, we have discretised the equations using central differences throughout. For the results presented in Section 4 we have set the outer boundary at $\eta_\infty = 2000$ with a step-length $\delta\eta = 0{\cdot}01$ in the η-direction. The initial time step $\delta\tau = 0{\cdot}001$ reducing to $0{\cdot}0001$ at $\tau = 0{\cdot}75$, to $0{\cdot}00002$ at $\tau = 0{\cdot}79895$, to $0{\cdot}00001$ at $\tau = 0{\cdot}79909$. The solution has been continued up to $\tau = 0{\cdot}79912$. From the checks we have carried out on grid sizes we are satisfied with the results presented below, subject to qualifications that we fully discuss.

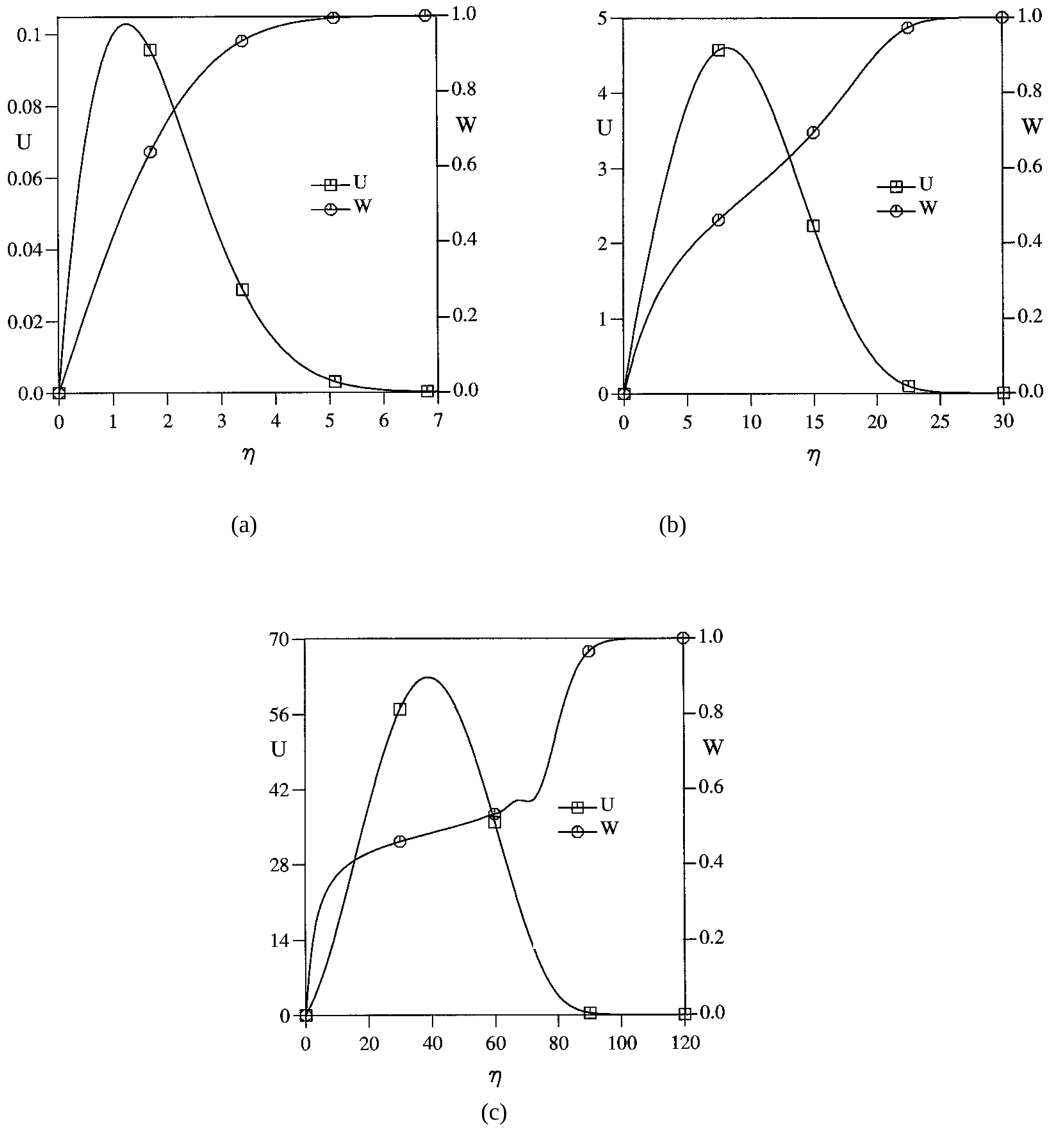

Figure 2. Computed profiles $U(\tau, \eta)$, $W(\tau, \eta)$ at various values of τ: (a) $\tau = 0{\cdot}4$, (b) $\tau = 0{\cdot}79$, (c) $\tau = 0{\cdot}7985$.

4. Results

Following the initial impulse, the axial boundary layer grows like the boundary layer on a flat plate, as in Equation (3.11). Only as the cross-flow boundary layer develops will this be modified. In Figure 2(a) we show the profiles U, W at $\tau = 0{\cdot}4$ ($t = 0{\cdot}6667$). Clearly, at this time, the cross-flow has had little influence on the axial velocity distribution given in (3.11). However, at $\tau = 0{\cdot}79$ ($t = 3{\cdot}7619$) we see, in Figure 2(b), that the axial profile has now developed a clear point of inflexion. The cross-flow is now more significant, and the boundary layer is growing in thickness at a much faster rate than can be accounted for by pure diffusion. As the outer part of the boundary layer is forced away from the boundary we may make a comparison with the work of Stewartson *et al.* [1]. In their study of the inlet flow to a loosely coiled pipe they also record a rapid thickening of the boundary layer at the inside bend of the pipe prior to the development of a singularity which heralds the eruption of fluid from the boundary layer to the core. However, there is a significant difference between the two solutions. In the present study the axial shear stress at $\alpha = \pi$, $\partial W/\partial\zeta|_{\zeta=0}$, remains finite as the eruptive singularity is approached. But in [1] that quantity approaches zero, suggesting some form of streamwise separation. Our unsteady problem is much more closely related to the work of Banks and Zaturska [13]. They consider the boundary layer on a sphere that is set into rotational motion, impulsively, about a fixed axis. The boundary layer that is formed on the sphere develops a singularity at the equator, which again may be interpreted as an eruption of fluid from that region. In that problem there is a well-established steady state in which fluid is flung from the equatorial region to form a swirling radial jet. Although the problem under consideration here is physically quite different from that in [13], the singular behaviour that is uncovered by Banks and Zaturska would appear to be fundamental, and applicable to unsteady flows which are essentially two-dimensional in nature, and in which there is a line onto which the flow is converging, a point that is also made in [10].

Before we examine the singular behaviour of our solution, and its implications, we consider another feature of the results. In Figure 2(c) we show the axial and cross-flow boundary-layer profiles at $\tau = 0{\cdot}7985$ ($t = 3{\cdot}9628$). We see that the axial profile has just developed a nonmonotonic behaviour. In [1] it is shown that for the related steady-flow problem such nonmonotonic behaviour is not possible. Nor is it in the azimuthal velocity profile of the rotating sphere problem [13]. For the present case, note that at neighbouring max, min of W we have $W_{\eta\eta} \gtreqless 0$ which implies, from (3.9), that $W_\tau \gtreqless 0$ which in turn precludes the possibility of such extrema.

We turn now to the nature of the singular behaviour of the solution. First we define axial and transverse displacement thicknesses as

$$\delta_1 = \int_0^\infty (1 - W)\,\mathrm{d}\zeta, \qquad \delta_2 = \int_0^\infty U\,\mathrm{d}\zeta. \tag{4.1}$$

As suggested by the results shown in Figure 2 both $\max(U) = U_m$, and the boundary-layer thickness, grow without bound as τ approaches some finite value τ_s, say. A close examination of our results suggests, as already intimated, that the emerging singularity is of the form discussed by Banks and Zaturska [13]. If that is the case, then the quantities U_m^{-1}, δ_1^{-2} and $\delta_2^{-\frac{2}{3}}$ will all vary linearly with t as $t \to t_s (\tau \to \tau_s)$. In Figure 3 we demonstrate clearly such linear behaviour, from which we have estimated $\tau_s = 0{\cdot}799147 (t_s = 3{\cdot}978776)$. However, as we have noted above, the results obtained for W are unreliable for $\tau > 0{\cdot}7985$, and our

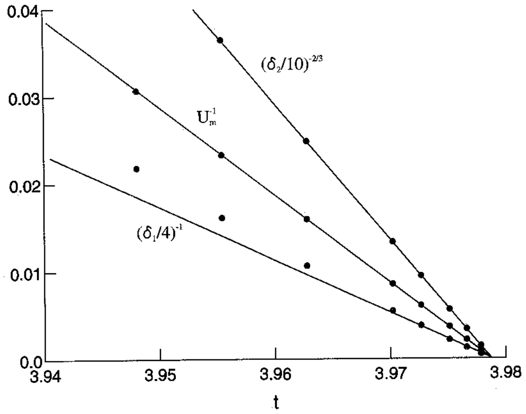

Figure 3. The variation with t, as $t \to t_s$ of $\delta_2^{-2/3}$, U_m^{-1} and δ_1^{-2}. The dots represent computed results, the straight lines have been drawn for comparison in each case.

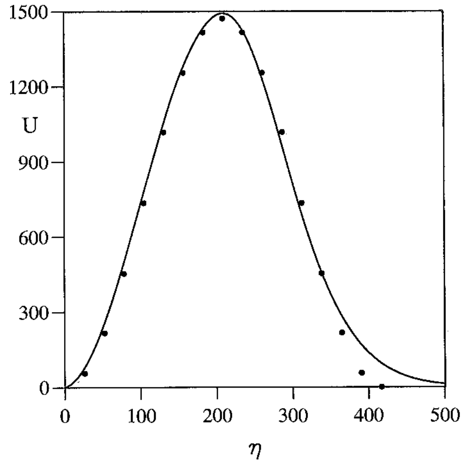

Figure 4. A comparison between the computed solution for U at $t = 3{\cdot}9781$ (full line), and the asymptotic solution (4.2a) (dots) with $H = H_0$ as in (4.4a) and $\beta = 0{\cdot}61$.

calculations have been continued up to $\tau = 0{\cdot}79912$ as displayed in Figure 3. Justification for including results for $\tau > 0{\cdot}7985$, certainly for U_m and δ_2, emerges when we look in more detail at the singular nature of the solution. The above results are in precise agreement with those of Lam [9] who, in addition, demonstrates from the complete boundary-layer solution that the transverse length scale of the eruption region shrinks like $(t_s - t)^{\frac{3}{2}}$ as $t \to t_s$.

To study the singular behaviour in more detail we follow Banks and Zaturska [13] and define new independent variables $\overline{\tau} = t_s - t$, $\overline{\eta} = \zeta\overline{\tau}^{\frac{1}{2}}$, and we then write, in Equations (3.8) and (3.9)

$$U = \overline{\tau}^{-1}\frac{\partial H}{\partial \overline{\eta}}, \qquad V = \overline{\tau}^{-\frac{3}{2}}H, \qquad W = G. \tag{4.2a, b, c}$$

If, for $\overline{\tau} \ll 1$, $H(\overline{\eta}, \overline{\tau}) = H_0(\overline{\eta}) + o(1)$, $G(\overline{\eta}, \overline{\tau}) = G_0(\overline{\eta}) + o(1)$, then the leading-order terms of the equations for H, G yield

$$(\tfrac{1}{2}\overline{\eta} - H_0)H_0'' + H_0'^2 - H_0' = 0, \qquad (\tfrac{1}{2}\overline{\eta} - H_0)G_0' = 0, \tag{4.3a, b}$$

where a prime denotes differentiation with respect to $\overline{\eta}$. These equations are essentially inviscid in nature, and have the solution

$$H_0 = \tfrac{1}{2}(\overline{\eta} - \beta^{-1}\sin\beta\overline{\eta}), \qquad G_0 = \gamma, \tag{4.4a, b}$$

which may be expected to hold in a central region of the boundary layer, away from its edges. At $\tau = 0{\cdot}7985$, which corresponds to $\overline{\tau} = 0{\cdot}016$, we already see in Figure 2(c) a plateau developing in the axial velocity consistent with constant G_0 in (4.4b). Furthermore, the terms that have been omitted in the transverse momentum equation, to yield (4.3a), are of relative order $\overline{\tau}^2$ compared with those retained, and so $O(10^{-4})$, and smaller, for $\tau \geqslant 0{\cdot}7985$. For that reason, in spite of anomalies that may emerge in W, we have confidence in the results presented in Figure 3 for U_m and δ_2. That δ_1 appears to be correctly behaved is perhaps fortuitous, implying that any oscillatory behaviour that develops in W does not change the axial mass flux in the boundary layer. We may estimate the constant β in (4.4) from the last computed velocity profile $U(\eta, \tau)$ by ensuring that $\overline{\eta} = \pi/\beta$ coincides with the value $\eta = \eta_m$ at which $U = U_m$. This gives $\beta \approx 0{\cdot}61$ which compares with the value 0·71 in the problem considered by Banks and Zaturska [13] for the rotating-sphere. In Figure 4 we compare the computed profile $U(\eta, \tau)$ at $\tau = 0{\cdot}79912$ with the asymptotic profile, using (4.4a). The good agreement adds confidence to our results overall.

5. Conclusions

In this paper we have considered the unsteady fully-developed flow in a loosely coiled pipe of circular cross-section, when a constant pressure gradient is impulsively applied at some initial instant. The time scale on which we have analysed the flow is one on which the axial core flow maintains a uniform value, but the flow in the boundary layer at the pipe wall develops significantly. In particular, there is a transverse cross flow in the boundary layer which transports fluid from the outer to the inner bend. At the inner bend, by symmetry, boundary layers impinge and our analysis there shows that a singularity develops at a finite time, which we interpret as an eruption of fluid from the boundary layer to the interior, or core flow. Thereafter changes to the core flow will take place that are beyond the scope of our analysis, until a fully developed steady state is reached. The study complements an earlier one by Stewartson *et al.* [1]. In [1] the authors consider the steady entry flow to the coiled pipe on an axial length scale over which the core flow is uniform, but again significant developments in the boundary layer take place. As in the present study, there is transport of fluid to the inner bend in the transverse boundary-layer flow which ultimately erupts into the interior, heralded by a singularity in the solution. On a longer length scale the core flow will undergo significant changes. In spite of the similarity between the two cases, there are significant differences associated with the structure of the singularity. The terminal stage of the unsteady flow discussed here is more closely related to the boundary layer that erupts from the equator of a rotating sphere, started from rest, as we have demonstrated.

Although both the present study, and the related one [1], only deal with the initial stages in which the core flow is unchanging, it is reasonable to ask if they shed any light on the steady fully-developed flow in a loosely coiled pipe. Before commenting further on that, we draw attention to a related problem. Lyne [14] has considered the unsteady flow in a coiled pipe induced by a small-amplitude oscillatory pressure gradient with zero mean. There is a second-order time-averaged flow which, for a large suitably defined Reynolds number analogous to the Dean number, consists of thin boundary layers on the pipe wall which collide, at the outside bend in this case, to form a thin viscous jet along the equator. These viscous layers surround effectively inviscid, semi-circular regions of counter-rotating flow with uniform vorticity. Lyne's work is fully confirmed by the Navier-Stokes solutions of Haddon [15]. This problem, as that of the steady fully-developed flow, is unusual insofar as the boundary layers exert a controlling influence on the core flow. However, only a single parameter has to be determined in that case, namely the magnitude of the vorticity in the recirculating regions. For the steady flow we are commenting on we have, see Section 2, $w_c = f(x)$, which is a more complex situation.

Returning to the steady fully-developed flow in a curved pipe, if $f(x)$ is a monotonically increasing function of x, and high Dean number solutions [4] of the full Equations (2.1) and (2.2) indicate this to be the case, we observe that the transverse boundary layer initially entrains fluid but ultimately, for some $\alpha > \frac{1}{2}\pi$, loses fluid as the boundary-layer fluid decelerates. There are then three possible scenarios. First, the transverse boundary layer 'empties' before the inner bend is reached, resulting in some form of flow separation. This would lead, in turn, to a gross distortion of the core flow, and accurate high Dean number solutions [4] of the full equations reveal no such phenomenon. Second, the boundary layers carry momentum up to the inner bend, where there is a collision which results in a thin viscous jet along the equator. This scenario is appealing in view of both the present study, and that of Stewartson *et al.*, where changes to the uniform core flow are initiated by an eruption of fluid, which may be interpreted in terms of a boundary-layer collision, at the inside bend, and the work of Lyne [14]. However, in the high-Dean-number solutions [4] there is no evidence whatsoever of a viscous jet forming along the equator. The third, and final, scenario is that in which a delicate balance is struck, whereby the transverse boundary layer persists up to $\alpha = \pi$, at which point it has lost all momentum. No collision of the boundary layers can then take place. The high-Dean-number solutions of (2.1) and (2.2) are not at variance with this. However, despite intensive efforts by Dennis (private communication), it has not been possible to construct an entirely satisfactory solution of (2.11) to (2.13) for steady flow based on this scenario. Difficulties close to $\alpha = \pi$ are encountered, and these have been commented on by Dennis and Riley [8].

We conclude that the solution of the governing equations for steady flow, in the high-Dean-number limit, remains unresolved. And, in spite of the fact that the flow will become unstable, we submit that the laminar flow in this limit provides a worthwhile and stimulating challenge for the computational fluid dynamicist.

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